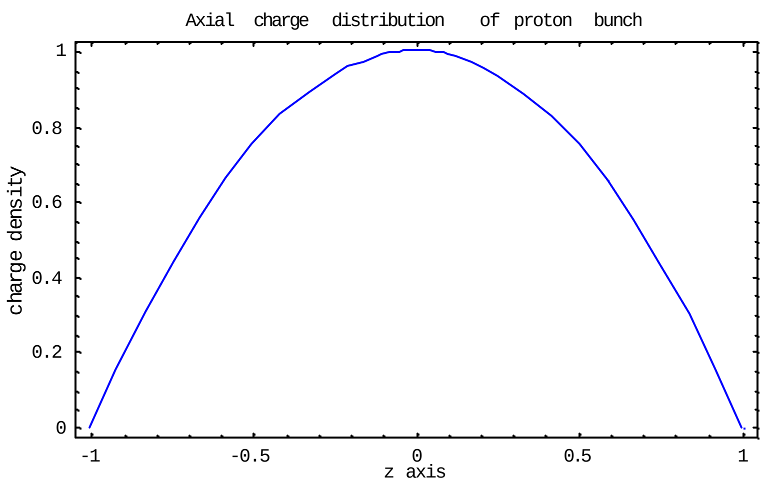
Longitudinal Emittance Formulas

AVT, 2-24-02

The following formulas are for the longitudinal emittance. I asked Vladimir Shiltsev for the definitions, and he sent me a note defining the emittance as the 100% area of a bucket for a beam with a parabolic beam distribution in the z direction. This is the equilibrium distribution that at least in first order the beam assumes.

The first relation connects the rms width of the pulse to the base width.



The base goes from –a to +a and the rms is given by:

Eq.1 $$s = \sqrt{5} a .$$

The total size of the rf bucket is given by:

Eq.2 $$A = \frac{16b}{w_{rf}} \sqrt{\frac{V_{rf} E_s}{2p\, h |h|}} .$$

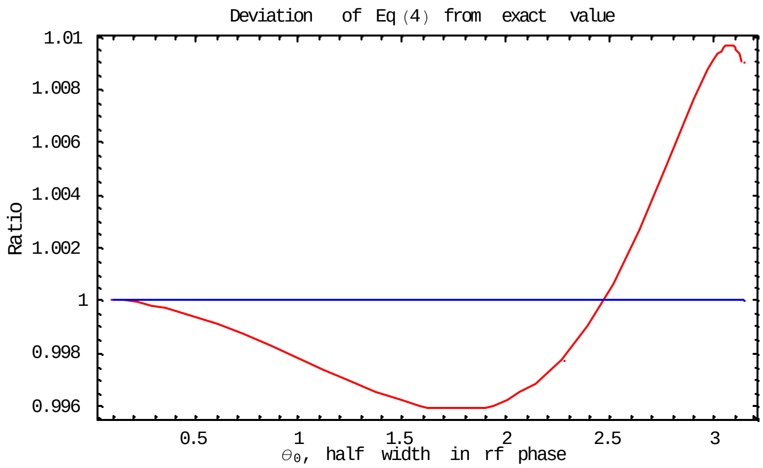
For small amplitude of oscillation, the area of a bucket with maximum phase excursion q_0 is given by :

Eq.3 $$e_z = \frac{16 A J_0^{\,2}}{p}$$

A more accurate expression that fits over the range of q_0 is given by:

Eq.4 $$e_z = \frac{16AJ_0^{\ 2}}{p}\left(1 - \frac{J_0^2}{20.2}\right).$$

A plot of the deviation of this formula from an accurate integration of the bucket area is shown below.



The fit is within 1% over the whole range. This is a small modification of a formula Vladimir gave me where he used 24 in place of 20.2. His value is an analytical one from taking the next order. Mine is an empirical fit.

Finally, one would like to use the rms pulse width. We substitute:

Eq. 5 $$q_0 = 2p\,\frac{\sqrt{5}s}{T_{rf}}$$

And get for 980 GeV and the time in ns:

Eq.6 $$e_z = 1.217s^{\ 2}(1 - .0276s^{\ 2}).$$